Homework 5 Oracle

MATH 220 Spring 2021

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93; 12021 H.E.

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Section 3.1

Problem 9

$$y'' + 3y' = 0, \quad y(0) = -2, \quad y'(0) = 3$$

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$

$$y = e^{rt} \quad \Longrightarrow \quad y' = re^{rt} \quad \Longrightarrow \quad y'' = r^2 e^{rt}$$

Substitute those expressions into the ODE

$$r^2 e^{rt} + 3(re^{rt}) = 0$$

Divide both sides by e^{rt}

$$r^2 + 3r = 0$$

Roots of this polynomial are $r_0 = -3$ and $r_1 = 0$. Two solutions to the ODE are $y = e^{-3t}$ and $y = e^0 = 1$. Therefore, the general solution is

$$y(t) = C_1 e^{-3t} + C_2$$

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Differentiating y gives us

$$y'(t) = -3C_1 e^{-3t}$$

Now, we can determine our constants by applying the two initial conditions we know

$$\begin{cases} y(0) = C_1 + C_2 = -2 \\ y'(0) = -3C_1 = 3 \end{cases}$$

Therefore $C_1 = -1$ and $C_2 = -1$, therefore

$$y(t) = -e^{-3t} - 1$$

This solution converges to -1 as $t \to \infty$.

Problem 13 [FOR GRADE]

Find a differential equation whose general solution is

$$y = c_1 e^{2t} + c_2 e^{-3t}$$

We see the roots are $r_0 = -3$ and $r_1 = 2$. Alternatively, you can make a set of solutions, and call it $r = \{-3, 2\}$. So

$$(r+3)(r-2) = 0$$
$$\implies r^2 + r - 6 = 0$$

Multiply both sides by e^{rt}

$$r^2 e^{rt} + r e^{rt} - 6e^{rt} = 0$$

Therefore, the differential equation is

$$y'' + y' - 6y = 0$$

Problem 16

This is a linear homogeneous constant-coefficient ODE, apply the same method as in Problem 9. Find that $r = \{-1, 2\}$ and the general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{2t}$$

The derivative would be

$$y'(t) = -C_1 e^{-t} + 2C_2 e^{2t}$$

Let us solve the initial conditions

$$\begin{cases} y(0) = C_1 + C_2 = \alpha \\ y'(0) = -C_1 + 2C_2 = 2 \end{cases} \implies \begin{cases} C_1 = \frac{2}{3}(\alpha - 1) \\ C_2 = \frac{1}{3}(\alpha + 2) \end{cases}$$

Therefore,

$$y(t) = \frac{2}{3}(\alpha - 1)e^{-t} + \frac{1}{3}(\alpha + 2)e^{2t}$$

We can see that if $t \to \infty$, then $y \to \infty$. Therefore, set $\alpha = -2$.

Problem 19

$$y'' + 5y' + 6y = 9, \quad y(0) = 2, \quad y'(0) = \beta,$$

where $\beta > 0$.

Part (a)

This is a linear homogeneous constant-coefficient ODE, find that $r = -\frac{1}{2}, \frac{1}{2}$. The two solutions are

$$y(t) = C_1 e^{-\frac{t}{2}} + C_2 e^{\frac{t}{2}}$$

Then

$$y'(t) = -\frac{C_1}{2}e^{-\frac{t}{2}} + \frac{C_2}{2}e^{\frac{t}{2}}$$

Solve

$$\begin{cases} y(0) = C_1 + C_2 = 2 \\ y'(0) = -\frac{C_1}{2} + \frac{C_2}{2} = \beta \end{cases} \implies \begin{cases} C_1 = 1 - \beta \\ C_2 = 1 + \beta \end{cases}$$

Finally,

$$y(t) = (1-\beta)e^{-\frac{t}{2}} + (1+\beta)e^{\frac{t}{2}}$$

To prevent the solution from going to the infinity and beyond, set $\beta = -1$.

Part (b, c, d)

See Professor Van Vleck's notes on this problem.

Problem 21 [FOR GRADE]

$$ay'' + by' + cy = 0,$$

where $a, b, c \in \mathbb{R}$ and $a > 0$.

This is yet again another linear homogeneous constant-coefficient ODE. Find that

$$a\left(r^2 e^{rt}\right) + b\left(re^{rt}\right) + c\left(e^{rt}\right) = 0$$

Divide both sides by e^{rt}

$$\begin{aligned} ar^2 + br + c &= 0 \\ \implies r &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

Part (a)

For the roots to be real, different and negative, $b > 0$ and $0 < c < \frac{b^2}{4a}$.

Part (b)

For the roots to be real with opposite signs, $c < 0$.

Part (c)

For the roots to be real, different and positive, $b < 0$ and $0 < c < \frac{b^2}{4a}$.

Section 3.2

Problem 5

The Wronskian of these two functions is

$$\begin{aligned}
W &= \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ \frac{d}{d\theta}\left(\cos^2\theta\right) & \frac{d}{d\theta}(1+\cos 2\theta) \end{vmatrix} \\
&= \begin{vmatrix} \cos^2\theta & 1+\cos 2\theta \\ 2\cos\theta(-\sin\theta) & -2\sin 2\theta \end{vmatrix} \\
&= \cos^2\theta(-2\sin 2\theta) - (1+\cos 2\theta)[2\cos\theta(-\sin\theta)] \\
&= -2\cos^2\theta\sin 2\theta + 2\sin\theta\cos\theta(1+\cos 2\theta) \\
&= -2\cos^2\theta(2\sin\theta\cos\theta) + 2\sin\theta\cos\theta\left(1+2\cos^2\theta - 1\right) \\
&= -4\cos^2\theta\sin\theta\cos\theta + 4\sin\theta\cos\theta\cos^2\theta \\
&= 0
\end{aligned}$$

Problem 22 [FOR GRADE]

$$y'' - y' - 2y = 0$$

Note: Solutions for this problem are based on Jock's solutions.

Part (a)

Calculate $W(y_1, y_2)$ the Wronskian of y_1 and y_2.

$$
\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} \\
&= e^{-t}\left(2e^{2t}\right) - e^{2t}\left(-e^{-t}\right) \\
&= 2e^{t} + e^{t} \\
&= 3e^{t}
\end{aligned}
$$

Since $W(y_1, y_2) \neq 0, y_1$ and y_2 form a fundamental set of solutions.

Part (b)

Check that y_3 is a solution of the ODE.

$$
\begin{aligned}
y_3'' - y_3' - 2y_3 &\stackrel{?}{=} 0 \\
\frac{d^2}{dt^2}\left(-2e^{2t}\right) - \frac{d}{dt}\left(-2e^{2t}\right) - 2\left(-2e^{2t}\right) &\stackrel{?}{=} 0 \\
\left(-8e^{2t}\right) - \left(-4e^{2t}\right) - 2\left(-2e^{2t}\right) &\stackrel{?}{=} 0 \\
-8e^{2t} + 4e^{2t} + 4e^{2t} &\stackrel{?}{=} 0 \\
0 &= 0
\end{aligned}
$$

Now check that $y_4 = e^{-t} + 2e^{2t}$ is a solution of the ODE.

$$
\begin{aligned}
y_4'' - y_4' - 2y_4 &\stackrel{?}{=} 0 \\
\frac{d^2}{dt^2}\left(e^{-t} + 2e^{2t}\right) - \frac{d}{dt}\left(e^{-t} + 2e^{2t}\right) - 2\left(e^{-t} + 2e^{2t}\right) &\stackrel{?}{=} 0 \\
\left(e^{-t} + 8e^{2t}\right) - \left(-e^{-t} + 4e^{2t}\right) - 2\left(e^{-t} + 2e^{2t}\right) &\stackrel{?}{=} 0 \\
e^{-\ell} + 8e^{2t} + e^{-} - 4e^{2t} - 2e^{-} - 4e^{2t} &\stackrel{?}{=} 0 \\
0 &= 0
\end{aligned}
$$

Now check that $y_5 = 2y_1(t) - 2y_3(t) = 2e^{-t} - 2\left(-2e^{2t}\right) = 2e^{-t} + 4e^{2t}$ is a solution of the ODE.

$$
\begin{aligned}
y_5'' - y_5' - 2y_5 &\stackrel{?}{=} 0 \\
\frac{d^2}{dt^2}\left(2e^{-t} + 4e^{2t}\right) - \frac{d}{dt}\left(2e^{-t} + 4e^{2t}\right) - 2\left(2e^{-t} + 4e^{2t}\right) &\stackrel{?}{=} 0 \\
\left(2e^{-t} + 16e^{2t}\right) - \left(-2e^{-t} + 8e^{2t}\right) - 2\left(2e^{-t} + 4e^{2t}\right) &\stackrel{?}{=} 0 \\
2e^{-} + 16e^{2t} + 2e^{-} - 8e^{2t} - 4e^{-} - 8e^{2t} &\stackrel{?}{=} 0 \\
0 &= 0
\end{aligned}
$$

Part (c)

Calculate $W(y_1, y_3)$, the Wronskian of y_1 and y_3.

$$\begin{aligned}
W(y_1, y_3) &= \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t} & -2e^{2t} \\ -e^{-t} & -4e^{2t} \end{vmatrix} \\
&= e^{-t}\left(-4e^{2t}\right) - \left(-2e^{2t}\right)\left(-e^{-t}\right) \\
&= -4e^{t} - 2e^{t} \\
&= -6e^{t}
\end{aligned}$$

Since $W(y_1, y_3) \neq 0, y_1$ and y_3 form a fundamental set of solutions.

Now calculate $W(y_2, y_3)$, the Wronskian of y_2 and y_3

$$\begin{aligned}
W(y_2, y_3) &= \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix} \\
&= \begin{vmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{vmatrix} \\
&= e^{2t}\left(-4e^{2t}\right) - \left(-2e^{2t}\right)\left(2e^{2t}\right) \\
&= -4e^{4t} + 4e^{4t} \\
&= 0
\end{aligned}$$

Since $W(y_2, y_3) = 0, y_2$ and y_3 do not form a fundamental set of solutions. Now calculate $W(y_1, y_4)$, the Wronskian of y_1 and y_4

$$\begin{aligned}
W(y_1, y_4) &= \begin{vmatrix} y_1 & y_4 \\ y_1' & y_4' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t} & e^{-t} + 2e^{2t} \\ -e^{-t} & -e^{-t} + 4e^{2t} \end{vmatrix} \\
&= e^{-t}\left(-e^{-t} + 4e^{2t}\right) - \left(e^{-t} + 2e^{2t}\right)\left(-e^{-t}\right) \\
&= -e^{-2t} + 4e^{t} + e^{-2t} + 2e^{t} \\
&= 6e^{t}
\end{aligned}$$

Since $W(y_1, y_4) \neq 0, y_1$ and y_4 form a fundamental set of solutions. Now calculate

$W(y_4, y_5)$, the Wronskian of y_4 and y_5.

$$\begin{aligned}
W(y_4,y_5) &= \begin{vmatrix} y_4 & y_5 \\ y_4' & y_5' \end{vmatrix} \\
&= \begin{vmatrix} e^{-t}+2e^{2t} & 2e^{-t}+4e^{2t} \\ -e^{-t}+4e^{2t} & -2e^{-t}+8e^{2t} \end{vmatrix} \\
&= \left(e^{-t}+2e^{2t}\right)\left(-2e^{-t}+8e^{2t}\right)-\left(2e^{-t}+4e^{2t}\right)\left(-e^{-t}+4e^{2t}\right) \\
&= -2e^{-2t}+8e^{t}-4e^{t}+16e^{4t}-\left(-2e^{-2t}+8e^{t}-4e^{t}+16e^{4t}\right) \\
&= 0
\end{aligned}$$

Since $W(y_4, y_5) = 0, y_4$ and y_5 do not form a fundamental set of solutions.

Problem 24

$$(\cos t)y'' + (\sin t)y' - ty = 0$$

Then

$$y'' + \frac{\sin t}{\cos t} - \frac{t}{\cos t}y = 0$$

so

$$p(t) = \tan t$$

Then

$$W = C\exp\left(-\int \tan t dt\right)$$

By Abel's Theorem

$$W = C\exp(\ln(\cos t)) \implies W = C \times \cos t$$

Problem 31

The equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is said to be exact if it can be written in the form

$$\big(P(x)y'\big)' + (f(x)y)' = 0$$

where $f(x)$ is to be determined in terms of $P(x), Q(x),$ and $R(x)$ The latter equation can be integrated once immediately, resulting in a first-order linear equation for y that can be solved as in Section 2.1. By equating the coefficients of the preceding equations and then eliminating $f(x)$, show that a necessary condition for exactness is

$$P''(x) - Q'(x) + R(x) = 0$$

It can be shown that this is also a sufficient condition.