Homework 11 Oracle

MATH 220 Spring 2021

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Solution to these problems are provided by STEM Jock.

Section 6.2

Problem 3

Statement

$$F(s) = \frac{2}{s^2+3s-4}$$

Solution

Factor the denominator.

$$\begin{aligned} F(s) &= \frac{2}{s^2+3s-4} \\ &= \frac{2}{(s+4)(s-1)} \\ &= \frac{2/5}{s-1} - \frac{2/5}{s+4} \end{aligned}$$

Take the inverse Laplace transform now to get $f(t)$.

$$f(t) = \frac{2}{5}e^t - \frac{2}{5}e^{-4t}$$

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Problem 11

Statement

$$y'' - 2y' + 4y = 0; \quad y(0) = 2, \quad y'(0) = 0$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' - 2y' + 4y\right\} = \mathcal{L}\{0\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} - 2\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = 0$$
$$\left[s^2Y(s) - sy(0) - y'(0)\right] - 2[sY(s) - y(0)] + 4Y(s) = 0$$

Plug in the initial conditions, $y(0) = 2$ and $y'(0) = 0$.

$$\left[s^2Y(s) - 2s\right] - 2[sY(s) - 2] + 4Y(s) = 0$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Solution Y , the transformed solution.

$$s^2Y(s) - 2sY(s) + 4Y(s) - 2s + 4 = 0$$
$$\left(s^2 - 2s + 4\right)Y(s) = 2s - 4$$

$$
\begin{aligned}
Y(s) &= \frac{2s-4}{s^2-2s+4} \\
&= \frac{2s-4}{s^2-2s+1+4-1} \\
&= \frac{2s-4}{(s-1)^2+3} \\
&= \frac{2s-2-4+2}{(s-1)^2+3} \\
&= \frac{2(s-1)-2}{(s-1)^2+3} \\
&= 2\frac{s-1}{(s-1)^2+3} - \frac{2}{(s-1)^2+3} \\
&= 2\frac{s-1}{(s-1)^2+3} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+3}
\end{aligned}
$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$.

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{2\frac{s-1}{(s-1)^2+3} - \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{(s-1)^2+3}\right\} \\
&= 2\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+3}\right\} - \frac{2}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s-1)^2+3}\right\} \\
&= 2e^t\cos\sqrt{3}t - \frac{2}{\sqrt{3}}e^t\sin\sqrt{3}t
\end{aligned}
$$

Problem 19

Statement

$$
y'' + y = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2, \\ 0, & 2 \le t < \infty \end{cases} \qquad y(0) = 0, \quad y'(0) = 0
$$

Solution

Let $f(t)$ represent the piecewise function on the right side.

$$
y'' + y = f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}
$$

Because this ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt.$$

Consequently, the first and second derivatives transform as follows.

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$$
$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$$

Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + y\right\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$
$$\left[s^2Y(s) - sy(0) - y'(0)\right] + Y(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Plug in the initial conditions, $y(0) = 0$ and $y'(0) = 0$, and $f(t)$.

$$\left[s^2Y(s)\right] + Y(s) = \int_0^1 e^{-st}(t)\,dt + \int_1^2 e^{-st}(2-t)\,dt + \int_2^\infty e^{-st}(0)\,dt$$

$$\begin{aligned}
\left(s^2+1\right)Y(s) &= \int_0^1 te^{-st}\,dt + 2\int_1^2 e^{-st}\,dt - \int_1^2 te^{-st}\,dt \\
&= \frac{1-(s+1)e^{-s}}{s^2} + 2\frac{e^{-s}-e^{-2s}}{s} - \frac{-e^{-2s} - 2se^{-2s} + (s+1)e^{-s}}{s^2} \\
&= \frac{1}{s^2} + \frac{e^{-2s}}{s^2} - \frac{2e^{-s}}{s^2}
\end{aligned}$$

Divide both sides by s^2+1.

$$\begin{aligned}
Y(s) &= \frac{1}{s^2\left(s^2+1\right)} + \frac{e^{-2s}}{s^2\left(s^2+1\right)} - \frac{2e^{-s}}{s^2\left(s^2+1\right)} \\
&= \frac{1}{s^2} - \frac{1}{s^2+1} + \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-2s} - 2\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-s}
\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to recover $y(t)$. Note that $H(t)$ is the Heaviside function, which is defined to be 1 if $t > 0$ and 0 if $t < 0$.

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}\{Y(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1} + \left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-2s} - 2\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} + \mathcal{L}^{-1}\left\{\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-2s}\right\} - 2\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)e^{-s}\right\} \\
&= (t - \sin t) + [(t-2) - \sin(t-2)]H(t-2) - 2[(t-1) - \sin(t-1)]H(t-1)
\end{aligned}
$$

Problem 22

Statement

$$
f(t) = te^{at} \tag{1}
$$

Solution

The Laplace transform of a function $f(t)$ is defined here as

$$
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt
$$

Substitute the given function and evaluate the integral.

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} te^{at}\,dt \\
&= \int_0^\infty \left(-\frac{\partial}{\partial s} e^{-st}\right) e^{at}\,dt \\
&= -\frac{d}{ds}\int_0^\infty e^{-st} e^{at}\,dt \\
&= -\frac{d}{ds}\int_0^\infty e^{(a-s)t}\,dt \\
&= -\frac{d}{ds}\left[\left.\frac{1}{a-s} e^{(a-s)t}\right|_0^\infty\right] \\
&= -\frac{d}{ds}\left(\frac{1}{s-a}\right) \\
&= -\left[-\frac{1}{(s-a)^2}\right] \\
&= \frac{1}{(s-a)^2}
\end{aligned}
$$

Section 6.3

Problem 7

Statement

$$f(t) = \begin{cases} 1, & 0 \leq t < 2 \\ e^{-(t-2)}, & t \geq 2 \end{cases}$$

Solution

Write $f(t)$ in terms of the Heaviside function, $H(t)$, which is defined to be 1 if $t > 0$ and 0 if $t < 0$.

$$\begin{aligned} f(t) &= 1[H(t) - H(t-2)] + e^{-(t-2)}H(t-2) \\ &= H(t) + \left[e^{-(t-2)} - 1\right] H(t-2) \\ &= u_0(t) + \left[e^{-(t-2)} - 1\right] u_2(t) \end{aligned}$$

Problem 13

Statement

$$F(s) = \frac{3!}{(s-2)^4}$$

Solution

Apply the two transforms,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{and} \quad \mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c),$$

together to solve this problem.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{3!}{(s-2)^4}\right\} \\ &= t^3 e^{2t} \end{aligned}$$

Problem 16

Statement

$$F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$$

Solution

Apply the two transforms,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{and} \quad \mathcal{L}\{f(t-c)H(t-c)\} = F(s)e^{-cs}$$

together to solve this problem.

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\{F(s)\} \\
&= \mathcal{L}^{-1}\left\{\frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-2s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-3s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s}e^{-4s}\right\} \\
&= (t-1)^0 H(t-1) + (t-2)^0 H(t-2) - (t-3)^0 H(t-3) - (t-4)^0 H(t-4) \\
&= H(t-1) + H(t-2) - H(t-3) - H(t-4) \\
&= u_1(t) + u_2(t) - u_3(t) - u_4(t)
\end{aligned}$$

Problem 20

Statement

$$F(s) = \frac{1}{9s^2 - 12s + 3}$$

Solution

Observe that the denominator can be written in terms of 3s.

$$F(s) = \frac{1}{(3s)^2 - 4(3s) + 3}$$

Factor the denominator.

$$F(s) = \frac{1}{[(3s) - 1][(3s) - 3]}$$

Partially decompose the fraction.

$$F(s) = \frac{-\frac{1}{2}}{(3s)-1} + \frac{\frac{1}{2}}{(3s)-3}$$

Apply the two transforms,

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a} \quad \text{and} \quad F(ks) = \mathcal{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\},$$

together to get $f(t)$

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} \\ &= -\frac{1}{2}\left(\frac{1}{3}e^{t/3}\right) + \frac{1}{2}\left(\frac{1}{3}e^{3t/3}\right) \\ &= -\frac{1}{6}e^{t/3} + \frac{1}{6}e^{t} \\ &= \frac{1}{6}\left(e^{t} - e^{t/3}\right) \end{aligned}$$

Section 6.4

Problem 1

Statement

$$y'' + y = f(t); \quad y(0) = 0, \quad y'(0) = 1; \quad f(t) = \begin{cases} 1, & 0 \le t < 3\pi \\ 0, & 3\pi \le t < \infty \end{cases}$$

Solution

Because the ODE is linear, the Laplace transform can be applied to solve it. The Laplace transform of a function $y(t)$ is defined here as

$$Y(s) = \mathcal{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)\,dt$$

Consequently, the first and second derivatives transform as follows. Apply the Laplace transform to both sides of the ODE.

$$\mathcal{L}\left\{y'' + y\right\} = \mathcal{L}\{f(t)\}$$

Use the fact that the transform is a linear operator.

$$\mathcal{L}\{y''\}+\mathcal{L}\{y\}=\mathcal{L}\{f(t)\}$$

$$\left[s^2Y(s)-sy(0)-y'(0)\right]+Y(s)=\int_0^{3\pi}e^{-st}(1)\,dt+\int_{3\pi}^{\infty}e^{-st}(0)\,dt$$

Plug in the initial conditions, $y(0)=0$ and $y'(0)=1$.

$$\left[s^2Y(s)-1\right]+Y(s)=\int_0^{3\pi}e^{-st}\,dt$$

As a result of applying the Laplace transform, the ODE has reduced to an algebraic equation for Y, the transformed solution.

$$\left(s^2+1\right)Y(s)-1=\left(-\tfrac{1}{s}e^{-st}\right)\Big|_0^{3\pi}$$

$$\left(s^2+1\right)Y(s)=\tfrac{1}{s}-\tfrac{1}{s}e^{-3\pi s}+1$$

$$Y(s)=\tfrac{1}{s(s^2+1)}-\tfrac{1}{s(s^2+1)}e^{-3\pi s}+\tfrac{1}{s^2+1}$$

$$=\left(\tfrac{1}{s}-\tfrac{s}{s^2+1}\right)-\left(\tfrac{1}{s}-\tfrac{s}{s^2+1}\right)e^{-3\pi s}+\tfrac{1}{s^2+1}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$
\begin{aligned}
y(t)&=\mathcal{L}^{-1}\{Y(s)\}\\
&=\mathcal{L}^{-1}\left\{\left(\frac{1}{s}-\frac{s}{s^2+1}\right)-\left(\frac{1}{s}-\frac{s}{s^2+1}\right)e^{-3\pi s}+\frac{1}{s^2+1}\right\}\\
&=(1-\cos t)-[1-\cos(t-3\pi)]H(t-3\pi)+\sin t\\
&=1+\sin t-\cos t-[1-\cos(t-\pi)]H(t-3\pi)\\
&=1+\sin t-\cos t-(1+\cos t)H(t-3\pi)\\
&=1+\sin t-\cos t-(1+\cos t)u_{3\pi}(t)
\end{aligned}
$$

Problem 2

Solution

Evaluate the inverse Laplace transforms.

In order to write $Y(s)$ in terms of known transforms, use partial fraction decomposition.

$$\frac{1}{s\left(s^2+2s+2\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+2s+2}$$

Multiply both sides by $s\left(s^2+2s+2\right)$.

$$1=A\left(s^2+2s+2\right)+(Bs+C)s$$

Plug in three random values of s to get a system of three equations for A, B, and $C .$

$s=0: \quad 1=2A$

$s=1: \quad 1=5A+B+C$

$s=2: \quad 1=10A+4B+2C$

Solving this system yields $A=1/2, B=-1/2$, and $C=-1$.

$$Y(s)=\left(\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{s^2+2s+2}\right)e^{-\pi s}-\left(\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{s^2+2s+2}\right)e^{-2\pi s}+\frac{1}{s^2+2s+2}$$

Complete the square in the denominators.

$$\begin{aligned}Y(s)&=\left(\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{s^2+2s+1+2-1}\right)e^{-\pi s}-\left(\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{s^2+2s+1+2-1}\right)e^{-2\pi s}+\frac{1}{s^2+2s+1+2-1}\\&=\left[\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{(s+1)^2+1}\right]e^{-\pi s}-\left[\frac{1/2}{s}+\frac{-\frac{1}{2}s-1}{(s+1)^2+1}\right]e^{-2\pi s}+\frac{1}{(s+1)^2+1}\end{aligned}$$

Make it so that $s+1$ appears in the numerators.

$$\begin{aligned}Y(s)&=\left[\frac{1/2}{s}+\frac{-\frac{1}{2}(s+1)-\frac{1}{2}}{(s+1)^2+1}\right]e^{-\pi s}-\left[\frac{1/2}{s}+\frac{-\frac{1}{2}(s+1)-\frac{1}{2}}{(s+1)^2+1}\right]e^{-2\pi s}+\frac{1}{(s+1)^2+1}\\&=\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s}\\&\quad-\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s}+\frac{1}{(s+1)^2+1}\end{aligned}$$

Take the inverse Laplace transform of $Y(s)$ now to get $y(t)$.

$$y(t)=\mathcal{L}^{-1}\{Y(s)\}$$

$$\begin{aligned}&=\mathcal{L}^{-1}\left\{\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s}\right.\\&\qquad\left.-\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s}+\frac{1}{(s+1)^2+1}\right\}\\&=\mathcal{L}^{-1}\left\{\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-\pi s}\right\}\\&\qquad-\mathcal{L}^{-1}\left\{\left[\frac{1/2}{s}-\frac{1}{2}\frac{s+1}{(s+1)^2+1}-\frac{1}{2}\frac{1}{(s+1)^2+1}\right]e^{-2\pi s}\right\}+\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}\end{aligned}$$

$$y(t) = \left[\frac{1}{2} - \frac{1}{2}e^{-(t-\pi)}\cos(t-\pi) - \frac{1}{2}e\right.$$

$$-\left[\frac{1}{2} - \frac{1}{2}e^{-(t-2\pi)}\cos(t-2\pi) - \frac{1}{2}e^{-(t-2\pi)}\sin(t-2\pi)\right]H(t-2\pi) + e^{-t}\sin t$$

$$= \left(\frac{1}{2} + \frac{1}{2}e^{\pi-t}\cos t + \frac{1}{2}e^{\pi-t}\sin t\right.$$

$$-\left(\frac{1}{2} - \frac{1}{2}e^{2\pi-t}\cos t - \frac{1}{2}e^{2\pi-t}\sin t\right)H(t-2\pi) + e^{-t}\sin t$$

$$= \frac{1}{2}\left(1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right)H(t-\pi)$$

$$-\frac{1}{2}\left(1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right)H(t-2\pi) + e^{-t}\sin t$$

Therefore,

$$y(t) = \frac{1}{2}\left(1 + e^{\pi-t}\cos t + e^{\pi-t}\sin t\right)u_{\pi}(t) - \frac{1}{2}\left(1 - e^{2\pi-t}\cos t - e^{2\pi-t}\sin t\right)u_{2\pi}(t) + e^{-t}\sin t$$